Aufbau der natürlichen, ganzen, rationalen und surrealen Zahlen

Ingo Blechschmidt

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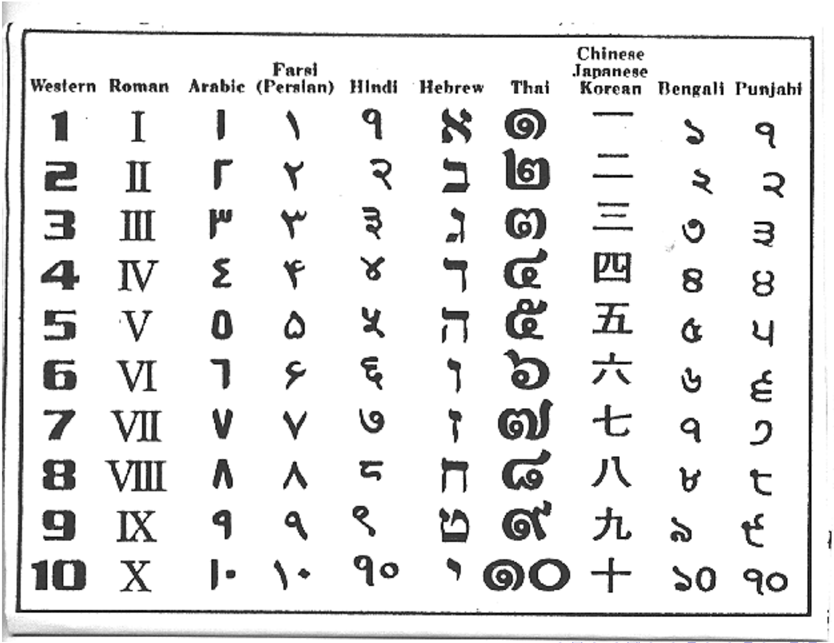


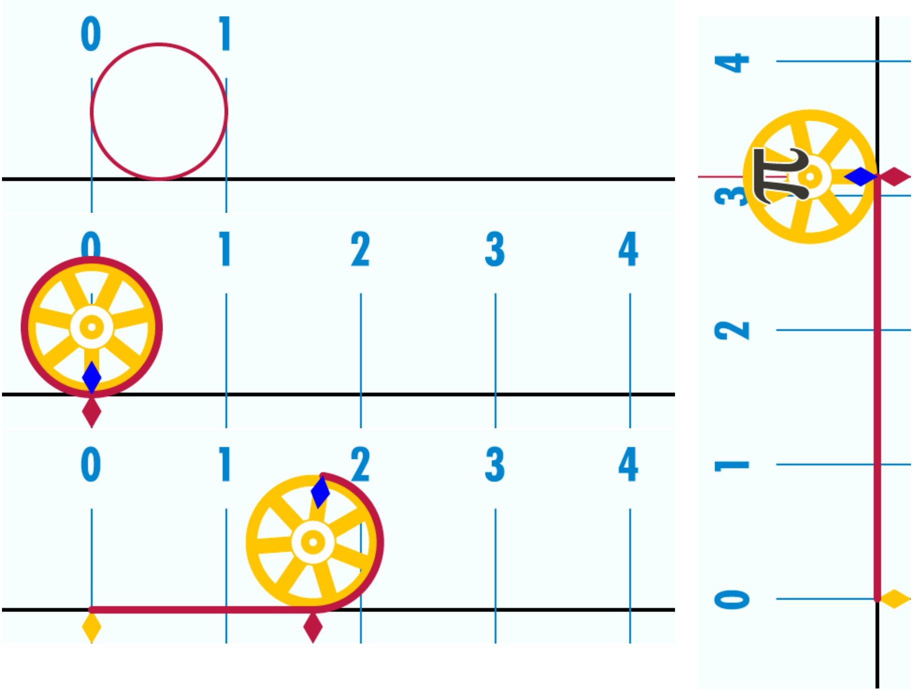
Western	Roman	Arabic	Farsi (Persian)	Hindi	Hebrew	Thai	Chinese Japanese Korean	Bengali	Punjabi
1	I	١	۱	१	א	๑	一	১	੧
2	II	٢	۲	२	ב	๒	二	২	੨
3	III	٣	۳	३	ג	๓	三	৩	੩
4	IV	٤	۴	४	ד	๔	四	৪	੪
5	V	٥	۵	५	ה	๕	五	৫	੫
6	VI	٦	۶	६	ו	๖	六	৬	੬
7	VII	٧	۷	७	ז	๗	七	৭	੭
8	VIII	٨	۸	८	ח	๘	八	৮	੮
9	IX	٩	۹	९	ט	๙	九	৯	੯
10	X	١٠	۱۰	१०	י	๑๐	十	১০	੧੦

1

0

$$\pi \qquad e$$

$$e^{i\pi} + 1 = 0$$


0
1
0
1
2
3
4
0
1
2
3
4
4
3
2
1
0
π

Inhalt

$$2 + 2 = 4$$

allgemein anerkannt

Grauer Alltag

Wieso?

formalisieren

$$\boxed{\Gamma \vdash_{\text{TY}} \sigma : \kappa}$$

$$(\text{TyVar})\ \frac{d : \kappa \in \Gamma \quad \Gamma \vdash_k \kappa : \text{TY}}{\Gamma \vdash_{\text{TY}} d : \kappa} \qquad (\text{TyApp})\ \frac{\Gamma \vdash_{\text{TY}} \sigma_1 : \kappa_1 \to \kappa_2 \quad \Gamma \vdash_{\text{TY}} \sigma_2 : \kappa_1}{\Gamma \vdash_{\text{TY}} \sigma_1\ \sigma_2 : \kappa_2}$$

$$(\text{TySCon})\ \frac{(S_n : \overline{\kappa}^n \to \iota) \in \Gamma \quad \Gamma \vdash_{\text{TY}} \overline{\sigma : \kappa}^n}{\Gamma \vdash_{\text{TY}} S_n\ \overline{\sigma}^n : \iota} \qquad (\text{TyAll})\ \frac{\Gamma, a : \kappa \vdash_{\text{TY}} \sigma : \star \quad \Gamma \vdash_k \kappa : \delta \quad a \notin fv(\Gamma)}{\Gamma \vdash_{\text{TY}} \forall a{:}\kappa.\sigma : \star}$$

$$\boxed{\Gamma \vdash_{\text{CO}} \gamma : \sigma \sim \tau}$$

$$(\text{CoRefl})\ \frac{a{:}\kappa \in \Gamma \quad \Gamma \vdash_k \kappa : \text{TY}}{\Gamma \vdash_{\text{CO}} a : a \sim a} \qquad (\text{CoVar})\ \frac{g{:}\sigma \sim \tau \in \Gamma}{\Gamma \vdash_{\text{CO}} g : \sigma \sim \tau}$$

$$(\text{CoAllT})\ \frac{\begin{array}{c}\Gamma, a{:}\kappa \vdash_{\text{CO}} \gamma : \sigma \sim \tau \\ \Gamma \vdash_k \kappa : \text{TY} \quad a \notin fv(\Gamma)\end{array}}{\Gamma \vdash_{\text{CO}} \forall a{:}\kappa.\gamma : \forall a{:}\kappa.\sigma \sim \forall a{:}\kappa.\tau} \qquad (\text{CoInstT})\ \frac{\begin{array}{c}\Gamma \vdash_{\text{CO}} \gamma : \forall a{:}\kappa.\sigma \sim \forall b{:}\kappa.\tau \\ \Gamma \vdash_{\text{TY}} v : \kappa\end{array}}{\Gamma \vdash_{\text{CO}} \gamma @ v : [v/a]\sigma \sim [v/b]\tau}$$

$$(\text{SComp})\ \frac{\Gamma \vdash_{\text{CO}} \overline{\gamma : \sigma \sim \tau}^n \quad \Gamma \vdash_{\text{TY}} S_n\ \overline{\sigma}^n : \kappa}{\Gamma \vdash_{\text{CO}} S_n\ \overline{\gamma}^n : S_n\ \overline{\sigma}^n \sim S_n\ \overline{\tau}^n} \qquad (\text{Sym})\ \frac{\Gamma \vdash_{\text{CO}} \gamma : \sigma \sim \tau}{\Gamma \vdash_{\text{CO}} \text{sym}\ \gamma : \tau \sim \sigma} \qquad (\text{Trans})\ \frac{\begin{array}{c}\Gamma \vdash_{\text{CO}} \gamma_1 : \sigma_1 \sim \\ \Gamma \vdash_{\text{CO}} \gamma_2 : \sigma_2 \sim\end{array}}{\Gamma \vdash_{\text{CO}} \gamma_1 \circ \gamma_2 : \sigma}$$

$$(\text{Comp})\ \frac{\begin{array}{c}\Gamma \vdash_{\text{CO}} \gamma_1 : \sigma_1 \sim \tau_1 \quad \Gamma \vdash_{\text{CO}} \gamma_2 : \sigma_2 \sim \tau_2 \\ \Gamma \vdash_{\text{TY}} \sigma_1\ \sigma_2 : \kappa\end{array}}{\Gamma \vdash_{\text{CO}} \gamma_1\ \gamma_2 : \sigma_1\ \sigma_2 \sim \tau_1\ \tau_2} \qquad (\text{Left})\ \frac{\Gamma \vdash_{\text{CO}} \gamma : \sigma_1\ \sigma_2 \sim \tau_1\ \tau_2}{\Gamma \vdash_{\text{CO}} \text{left}\ \gamma : \sigma_1 \sim \tau_1} \qquad (\text{Right})\ \frac{\Gamma \vdash_{\text{CO}} \gamma : \sigma_1\ \sigma_2 \sim}{\Gamma \vdash_{\text{CO}} \text{right}\ \gamma : \sigma_2}$$

$$\text{(CompC)}\ \frac{\Gamma \vdash_{co} \gamma : \kappa_1 \sim \kappa_2 \quad \Gamma \vdash_{co} \gamma' : \sigma_1 \sim \sigma_2 \quad \Gamma \vdash_k \kappa_1 : \textsc{co}}{\Gamma \vdash_{co} \gamma \Rightarrow \gamma' : (\kappa_1 \Rightarrow \sigma_1) \sim (\kappa_2 \Rightarrow \sigma_2)} \quad \text{(LeftC)}\ \frac{\Gamma \vdash_{co} \gamma : \begin{matrix}\kappa_1 \Rightarrow \sigma_1 \\ \sim \\ \kappa_2 \Rightarrow \sigma_2\end{matrix}}{\Gamma \vdash_{co} \text{leftc}\ \gamma : \kappa_1 \sim \kappa_2} \quad \text{(RightC)}\ \frac{\Gamma \vdash_{co} \gamma : \begin{matrix}\kappa_1 \Rightarrow \\ \sim \\ \kappa_2 \Rightarrow\end{matrix}}{\Gamma \vdash_{co} \text{rightc}\ \gamma : \sigma}$$

$$(\sim)\ \frac{\Gamma \vdash_{co} \gamma_1 : \sigma_1 \sim \tau_1 \quad \Gamma \vdash_{co} \gamma_2 : \sigma_2 \sim \tau_2}{\Gamma \vdash_{co} \gamma_1 \sim \gamma_2 : (\sigma_1 \sim \sigma_2) \sim (\tau_1 \sim \tau_2)} \quad \text{(CastC)}\ \frac{\Gamma \vdash_{co} \gamma_1 : \kappa \quad \Gamma \vdash_{co} \gamma_2 : \kappa \sim \kappa'}{\Gamma \vdash_{co} \gamma_1 \blacktriangleright \gamma_2 : \kappa'}$$

$$\boxed{\Gamma \vdash_e e : \sigma}$$

$$\text{(Var)}\ \frac{u : \sigma \in \Gamma}{\Gamma \vdash_e u : \sigma} \quad \text{(Case)}\ \frac{\Gamma \vdash_e e : \sigma \quad \overline{\Gamma \vdash_p p \to e : \sigma \to \tau}}{\Gamma \vdash_e \mathbf{case}\ e\ \mathbf{of}\ \overline{p \to e} : \tau} \quad \text{(Let)}\ \frac{\Gamma \vdash_e e_1 : \sigma_1 \quad \Gamma, x : \sigma_1 \vdash_e e_2 :}{\Gamma \vdash_e \mathbf{let}\ x{:}\sigma_1 = e_1\ \mathbf{in}\ e_2 : \sigma}$$

$$\text{(Cast)}\ \frac{\Gamma \vdash_e e : \sigma \quad \Gamma \vdash_{co} \gamma : \sigma \sim \tau}{\Gamma \vdash_e e \blacktriangleright \gamma : \tau} \quad \text{(Abs)}\ \frac{\Gamma \vdash_{TY} \sigma_x : \star \quad \Gamma, x : \sigma_x \vdash_e e : \sigma}{\Gamma \vdash_e \lambda x{:}\sigma_x . e : \sigma_x \to \sigma} \quad \text{(App)}\ \frac{\Gamma \vdash_e e_1 : \sigma_2 \to \sigma_1 \quad \Gamma \vdash_e e_2 : \sigma_2}{\Gamma \vdash_e e_1\ e_2 : \sigma_1}$$

$$\text{(AbsT)}\ \frac{\Gamma, a : \kappa \vdash_e e : \sigma \quad \Gamma \vdash_k \kappa : \delta \quad a \notin fv(\Gamma)}{\Gamma \vdash_e \Lambda a{:}\kappa . e : \forall a{:}\kappa . \sigma} \quad \text{(AppT)}\ \frac{\Gamma \vdash_e e : \forall a{:}\kappa . \sigma \quad \Gamma \vdash_k \kappa : \delta \quad \Gamma \vdash_\delta \varphi : \kappa}{\Gamma \vdash_e e\ \varphi : \sigma[\varphi/a]}$$

$$\boxed{\Gamma \vdash_p p \to e : \sigma \to \tau}$$

$$\text{(Alt)}\ \frac{K : \forall \overline{a{:}\kappa} . \forall \overline{b{:}\iota} . \overline{\sigma} \to T\ \overline{a} \in \Gamma \quad \theta = [\overline{v/a}] \quad \Gamma, \overline{b{:}\theta(\iota)}, \overline{x{:}\theta(\sigma)} \vdash_e e : \tau}{\Gamma \vdash_p K\ \overline{b{:}\theta(\iota)}\ \overline{x{:}\theta(\sigma)} \to e : T\ \overline{v} \to \tau}$$

$$\boxed{\Gamma \vdash decl : \Gamma'} \qquad \boxed{\Gamma \vdash pgm : \sigma}$$

$$\text{(Data)}\ \frac{\overline{\Gamma \vdash_{TY} \sigma : \star} \quad \Gamma \vdash_k \kappa : \textsc{ty}}{\Gamma \vdash (\mathbf{data}\ T{:}\kappa\ \mathbf{where}\ \overline{K{:}\sigma}) : (T{:}\kappa, \overline{K{:}\sigma})} \quad \text{(Pgm)}\ \frac{\overline{\Gamma \vdash decl : \Gamma_d} \quad \Gamma = \Gamma \quad \Gamma \vdash e : \sigma}{\Gamma_0 \vdash \overline{decl}; e : \sigma}$$

$$\text{(Type)}\ \frac{\Gamma \vdash_k \kappa : \textsc{ty}}{\Gamma \vdash (\mathbf{type}\ S : \kappa) : (S{:}\kappa)} \quad \text{(Coerce)}\ \frac{\Gamma \vdash_k \kappa : \textsc{co}}{\Gamma \vdash (\mathbf{axiom}\ C : \kappa) : (C{:}\kappa)}$$

(oje)

(warum sitz' ich hier?)

(wär' ich doch nur
in die Kunstvor-
stellung gegangen. . .)

Natürliche Zahlen

- Definition eines Anfangselements 0
- Definition einer Nachfolgerfunktion S
- Definition der Zahlensymbole

$$
\begin{aligned}
1 &:= S(0) \\
2 &:= S(1) = S(S(0)) \\
3 &:= S(2) = S(S(S(0))) \\
4 &:= S(3) = S(S(S(S(0)))) \\
5 &:= S(4) = S(S(S(S(S(0))))) \\
6 &:= S(5) = S(S(S(S(S(S(0)))))) \\
&\vdots
\end{aligned}
$$

Addition

$$\begin{array}{lrcll} \text{A)} & n & + & 0 & := n \\ \text{B)} & n & + & S(m) & := S(n+m) \end{array}$$

Subtraktion

$$\begin{array}{lrcll} \text{A)} & n & - & 0 & := n \\ \text{B)} & S(n) & - & S(m) & := n - m \end{array}$$

Ganze Zahlen: Komplizierter Weg

3	-5	8
7	0	-2
4	-9	8

$+3$	-5	$+8$
$+7$	0	-2
$+4$	-9	$+8$

Komplizierter Weg

Feststellung: Jede ganze Zahl ist entweder. . .

- negativ,
- Null oder
- positiv.

Addition

$$\begin{aligned} 5 - 8 &= -(8-5) \\ 8 - 5 &= 8-5 \\ (-5) + 8 &= 8-5 \\ 5 + (-8) &= -(8-5) \\ (-5) + (-8) &= -(5+8) \end{aligned}$$

verwirrend!

Mathe ist doof!

Zum Glück: Einfacher Weg

~~Einfacher~~ Weg

Anderer Weg

Einfacher Weg

- Ganze Zahlen als *Maß für den Unterschied* zwischen zwei natürlichen Zahlen

- $5 - 3 = 6 - 4 = 7 - 5 = 8 - 6 = \cdots$
- $2 - 6 = 3 - 7 = 4 - 8 = 5 - 9 = \cdots$

Formalisierung

Ganze Zahlen als Paare zweier natürlicher Zahlen

$$\begin{aligned}
0_{\mathbb{Z}} &:\equiv (0,0) \equiv (1,1) \equiv (2,2) \equiv \cdots \\
1_{\mathbb{Z}} &:\equiv (1,0) \equiv (2,1) \equiv (3,2) \equiv \cdots \\
2_{\mathbb{Z}} &:\equiv (2,0) \equiv (3,1) \equiv (4,2) \equiv \cdots \\
(-1)_{\mathbb{Z}} &:\equiv (0,1) \equiv (1,2) \equiv (2,3) \equiv \cdots \\
(-2)_{\mathbb{Z}} &:\equiv (0,2) \equiv (1,3) \equiv (2,4) \equiv \cdots \\
&\vdots
\end{aligned}$$

Addition

$$(n, m) +_{\mathbb{Z}} (\nu, \mu) := (n +_{\mathbb{N}_0} \nu, m +_{\mathbb{N}_0} \mu)$$

Fragen

Fragen?

Bildquellen

- http://www.psteinbrenner.de/zahlen.gif
- http://upload.wikimedia.org/wikipedia/commons/d/dd/Pi-Seattle.jpg
- http://upload.wikimedia.org/wikipedia/commons/2/2a/Pi-unrolled-720.gif
- http://www.simpsonstrivia.com.ar/simpsons-photos/wallpapers/professor-frink.gif
- http://research.microsoft.com/%7Esimonpj/papers/ext%2Df/fc-tldi.pdf (S. 5)