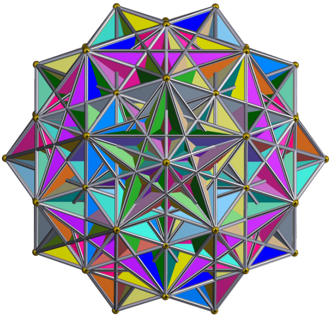


The curious world of four-dimensional geometry

Late Night Wondrous Mathematics at 37c3

Questions are welcome at any point! Don't save them until the end.

Ingo Blechschmidt

How does the sequence continue?

1,

How does the sequence continue?

$$1, \infty,$$

How does the sequence continue?

$1, \infty, 5,$

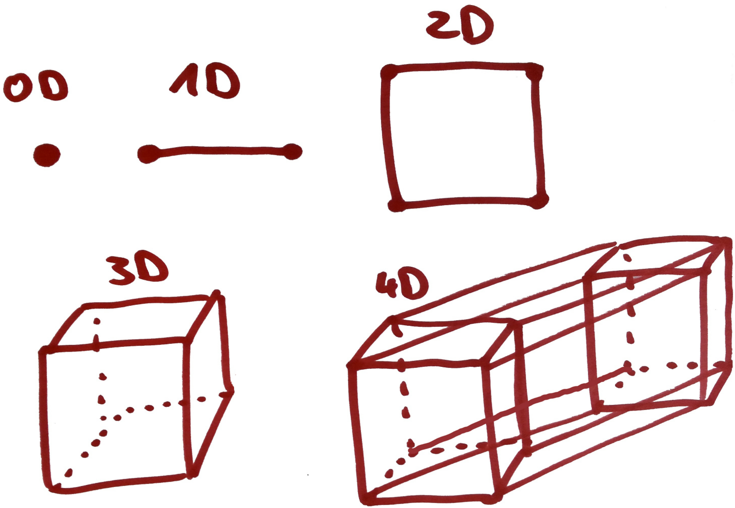
How does the sequence continue?

$1, \infty, 5, 6,$

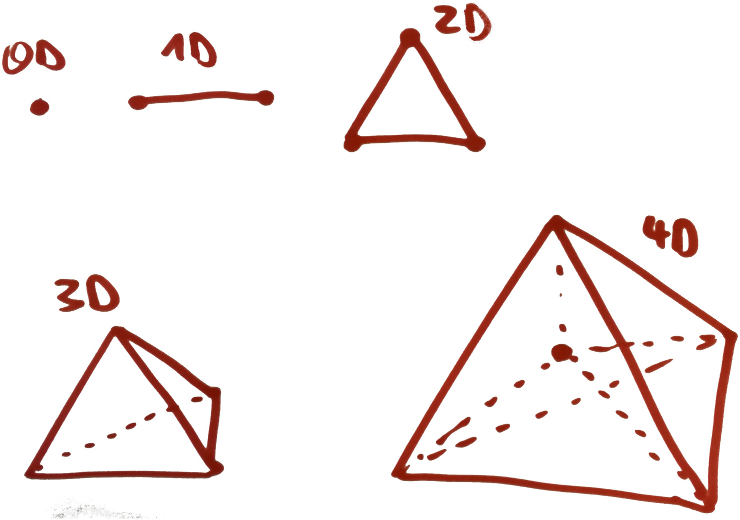
How does the sequence continue?

$1, \infty, 5, 6, ??$

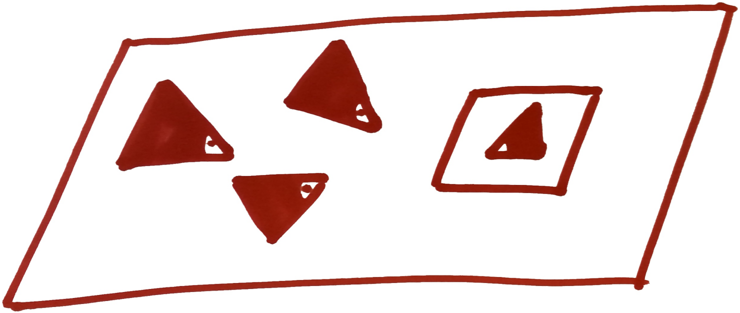
Four dimensions?



Four dimensions?

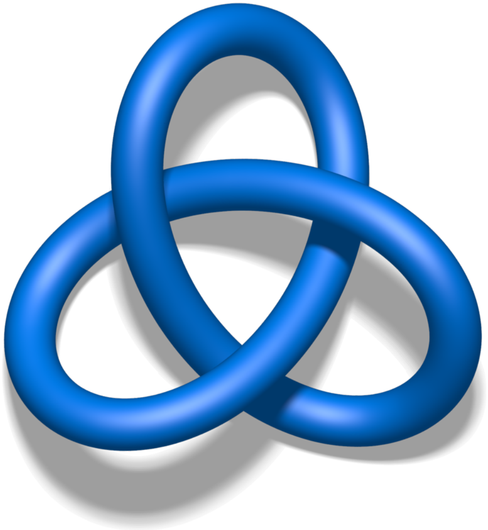


Four dimensions?



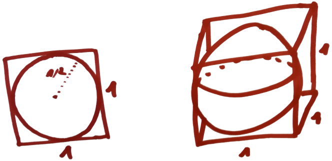
- On the previous slide you see two-dimensional projections of the three-dimensional cube and the four-dimensional hypercube (tesseract).
- We're talking about four spatial dimensions. This is not related to four-dimensional spacetime or eleven-dimensional string theory.
- A flatlander can be imprisoned by enclosing them with a square. But we, as three-dimensional beings, can free them by grabbing them, lifting them up in the third dimension, moving them a little to the side, and putting them back into flatland.
- Similarly, a four-dimensional being could free us if we were imprisoned in a three-dimensional cube.

Tying your shoelaces



- You can untie any knot in four dimensions. Two linked one-dimensional strings can always be separated in four dimensions.
- But it's possible to tangle an one-dimensional string with the two-dimensional surface of a sphere in four dimensions.
- More generally, in n dimensions, one can tangle a-dimensional objects with b-dimensional objects provided that $a + b \geq n - 1$.

Hypervolume of hyperballs



dimension	hypervolume	
$n = 2$	$\pi/4$	$\approx 0.785\,\mathrm{m}^2$
$n = 3$	$\pi/6$	$\approx 0.524\,\mathrm{m}^3$
$n = 4$	$\pi^2/32$	$\approx 0.308\,\mathrm{m}^4$
$n = 5$	$\pi^2/60$	$\approx 0.164\,\mathrm{m}^5$
$n = 6$	$\pi^3/384$	$\approx 0.081\,\mathrm{m}^6$
$n = 7$	$\pi^3/840$	$\approx 0.037\,\mathrm{m}^7$
$n \to \infty$	$\to 0$	

- The portion of the n-dimensional unit hypercube which is occupied by the inscribed n-dimensional hyperball gets arbitrary small in sufficiently high dimensions.

- The volume of such a hyperball is the answer to the following question: What is the probability that we managed to hit the hyperball with an dart, provided that we managed to hit the enclosing hyperball?

- Wikipedia gives derivations for these formulas.

- You can use the *power of negative thinking* to motivate that the formula for the n-dimensional volume of the n-dimensional hyperball does *not* contain π^n (but rather $\pi^{\lfloor n/2 \rfloor}$): Think about the zero- and one-dimensional case.

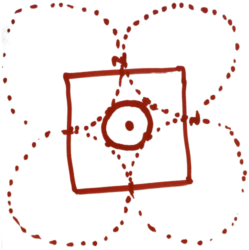
  A zero-dimensional ball is just a point. Its zero-dimensional volume is 1.

  An one-dimensional ball is just a line segment. Its one-dimensional volume is its length.

Love is important.

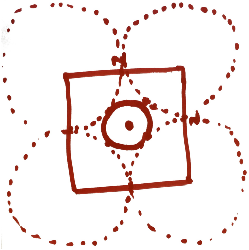
♡

Kissing hyperspheres



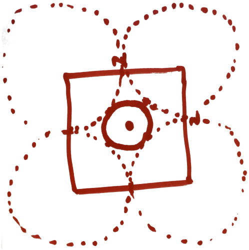
dimension	radius of the inner hypersphere
$n = 2$	

Kissing hyperspheres



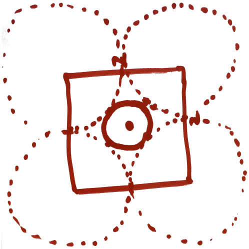
dimension	radius of the inner hypersphere
$n = 2$	$\sqrt{2} - 1$

Kissing hyperspheres



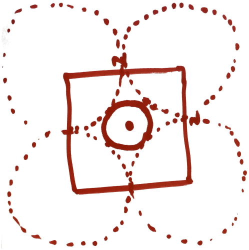
dimension	radius of the inner hypersphere
$n = 2$	$\sqrt{2} - 1$
$n = 3$	

Kissing hyperspheres



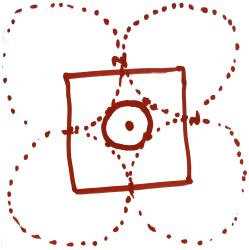
dimension	radius of the inner hypersphere
$n = 2$	$\sqrt{2} - 1$
$n = 3$	$\sqrt{3} - 1$

Kissing hyperspheres



dimension	radius of the inner hypersphere
$n = 2$	$\sqrt{2} - 1$
$n = 3$	$\sqrt{3} - 1$
$n = 4$	$\sqrt{4} - 1$

Kissing hyperspheres

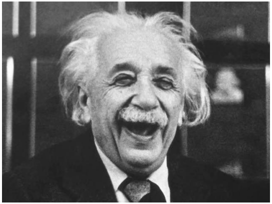


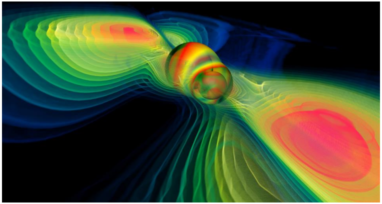
dimension	radius of the inner hypersphere
$n = 2$	$\sqrt{2} - 1$
$n = 3$	$\sqrt{3} - 1$
$n = 4$	$\sqrt{4} - 1$
n	$\sqrt{n} - 1$

The distance to the corners gets bigger and bigger.

- In two dimensions, the distance of a point (x, y) to the origin is $\sqrt{x^2 + y^2}$ (by the Pythagorean theorem).
- In three dimensions, the distance of a point (x, y, z) to the origin is $\sqrt{x^2 + y^2 + z^2}$.
- The pattern continues to arbitrary dimensions.
- In four dimensions, the “small hypersphere in the middle” has exactly the same size as the hyperspheres at the 16 vertices of the hypercube.
- In even greater dimensions, the hyperspheres at the vertices are so small that the “small hypersphere in the middle” is bigger than them and in fact bigger than the hypercube!

General relativity





Einstein's celebrated **field equation** states that

$$G = \kappa \cdot T,$$

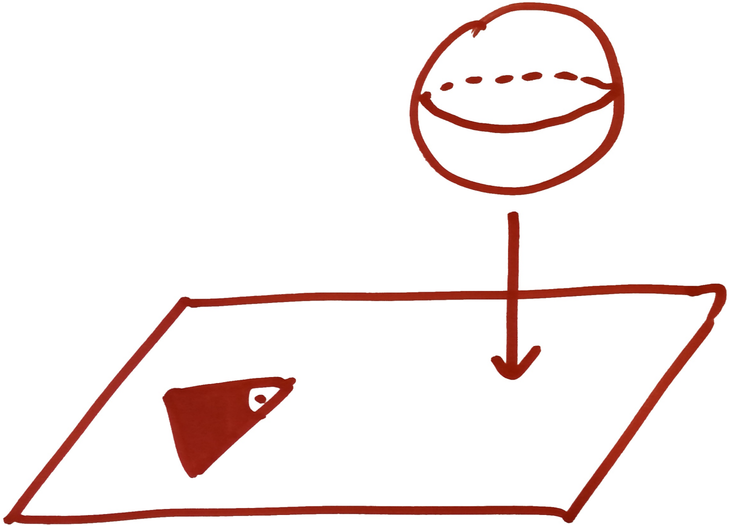
where

- G relates to the **curvature** of space,
- T measures the **mass distribution**, and
- κ is a constant.

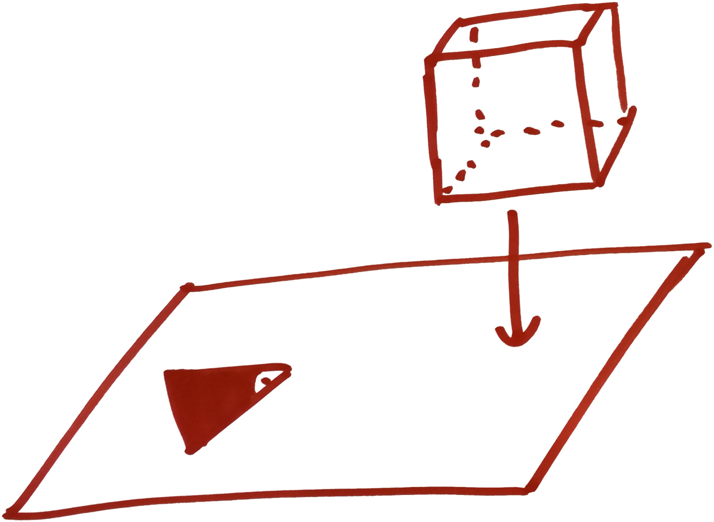
In $2+1$ dimensions, the equation implies $T = 0$. The theory is nontrivial only in four or more dimensions.

Details are in the article General relativity in two and three-dimensional space-times by Peter Collas.

A hyperball arrives



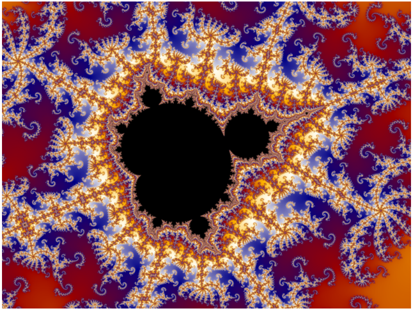
A tesseract arrives



A four-dimensional fractal

You know the Mandelbrot set. Maybe you also know the Julia sets associated to any point of the plane.

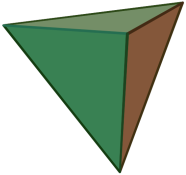
But did you know that these infinitely many fractals are just two-dimensional cuts of an unifying four-dimensional fractal? We invite you to play with it.



Platonic solids in 3d

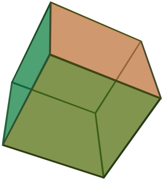
Tetrahedron

4 v, 6 e, 4 f



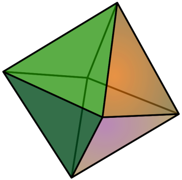
Hexahedron

8 v, 12 e, 6 f



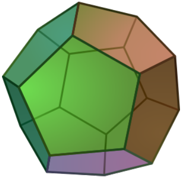
Octahedron

6 v, 12 e, 8 f



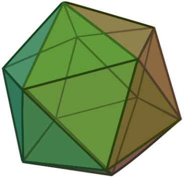
Dodecahedron

20 v, 30 e, 12 f



Icosahedron

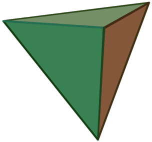
12 v, 30 e, 20 f



Platonic solids in 4d

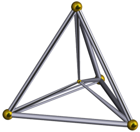
Tetrahedron

4v, 6e, 4f



Pentachoron

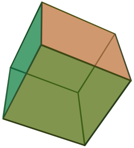
5v, 10e, 10f, 5c



Platonic solids in 4d

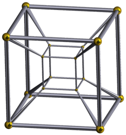
Hexahedron

8v, 12e, 6f



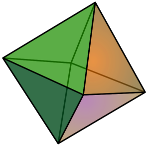
Octachoron

16v, 32e, 24f, 8c



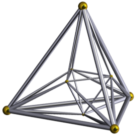
Octahedron

6v, 12e, 8f



Hexadecachoron

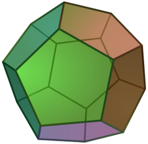
8v, 24e, 32f, 16c



Platonic solids in 4d

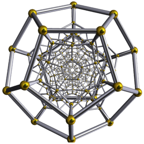
Dodecahedron

20v, 30e, 12f



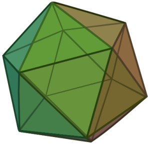
Hecatonicosachoron

600v, 1200e, 720f, 120c



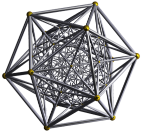
Icosahedron

12v, 30e, 20f



Hexacosichoron

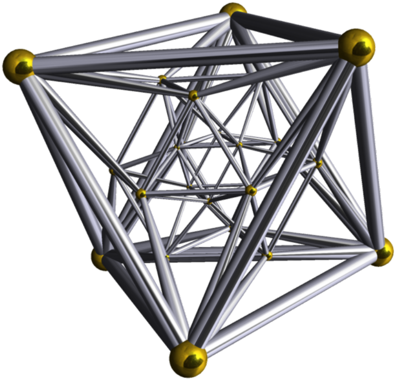
120v, 720e, 1200f, 600c



Platonic solids in 4d

Icositetrachoron

24v, 96e, 96f, 24c

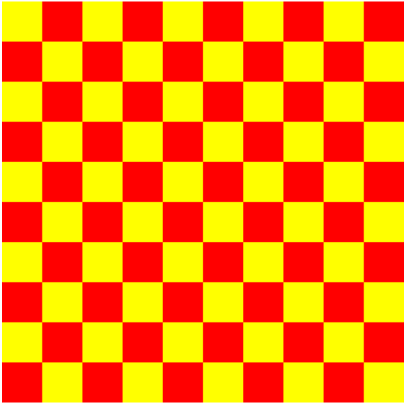


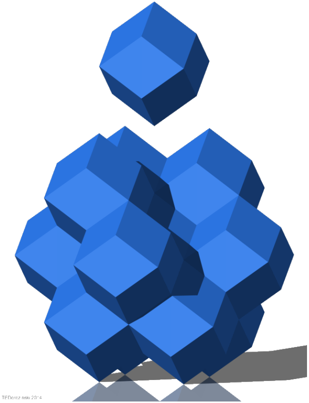
Platonic solids in 4d

Icositetrachoron

24v, 96e, 96f, 24c

Tesselation

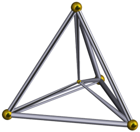




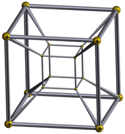
The 24-cell tesselates four-dimensional space.

Overview

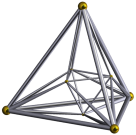
Pentachoron
5v, 10e, 10f, 5c



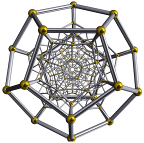
Octachoron
16v, 32e, 24f, 8c



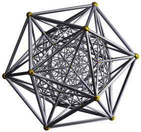
Hexadecachoron
8v, 24e, 32f, 16c



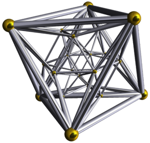
Hecatonicosachoron
600v, 1200e, 720f, 120c



Hexacosichoron
120v, 720e, 1200f, 600c



Icositetrachoron
24v, 96e, 96f, 24c



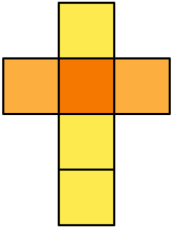
In arbitrary dimensions

dimension	number of Platonic solids
$n = 1$	1 (just the line segment)
$n = 2$	∞ (triangle, square, …; any regular polygon)
$n = 3$	5
$n = 4$	6
$n = 5$	3 (just the simplex, the hypercube and its dual)
$n = 6$	3 (just the simplex, the hypercube and its dual)
$n = 7$	3 (just the simplex, the hypercube and its dual)
and so on	

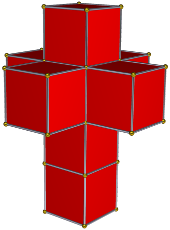
- The only platonic solid which can be used to tesselate three-dimensional space is the cube.
- In four dimensions, both the tesseract and the 24-cell work.
- This has a deeper reason: In any dimension n, the n-dimensional analogue of the rhombic dodecahedron can be used to tesselate n-dimensional space. In dimension $n = 3$ the rhombic dodecahedron is not a Platonic solid; in dimension $n = 4$ it is (and is also called the "24-cell").

Glueing four-dimensional shapes

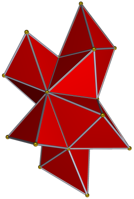
Cube



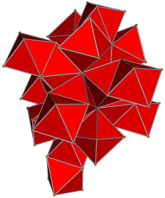
Tesseract

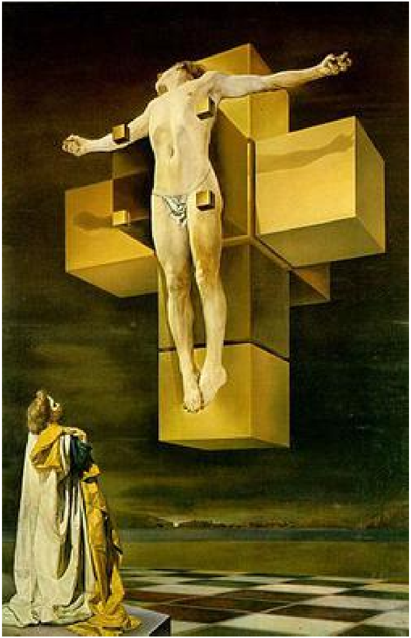


16-cell



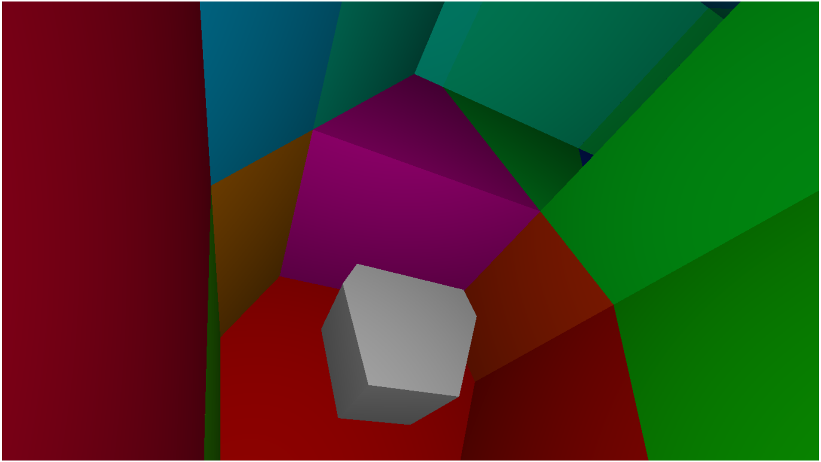
24-cell





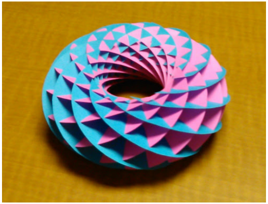
Salvador Dalí: **Corpus Hypercubus** (1954)

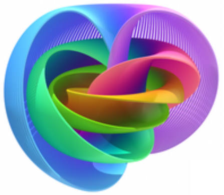
A four-dimensional labyrinth



The fourth dimension …

1. is intriguingly beautiful,
2. helps at understanding the third dimension,





3. is indispensible for modern physics,
4. is the only dimension which is still largely not understood.

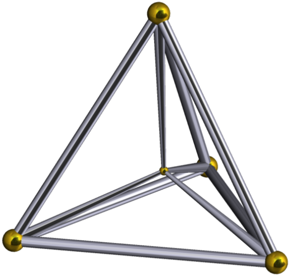
Dimension	1	2	3	4	5	6	7	8	9	…
Number of spheres	1	1	1	??	1	1	28	2	8	…

Catharina Stroppel
knot theorist

Julia Grigsby
low-dimensional topologist

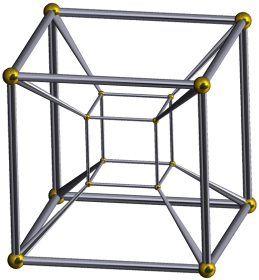
Applause for our heros!

Pentachoron

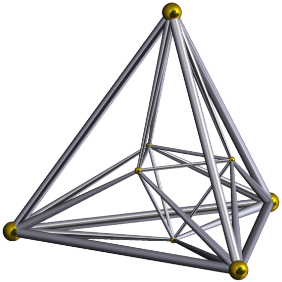


Applause for our heros!

Tesserakt

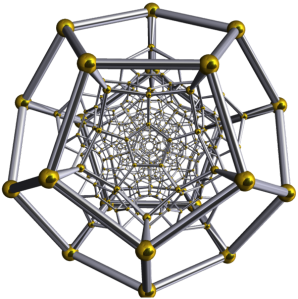


Hexadecachoron

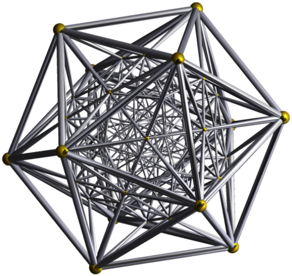


Applause for our heros!

Hecatonicosachoron

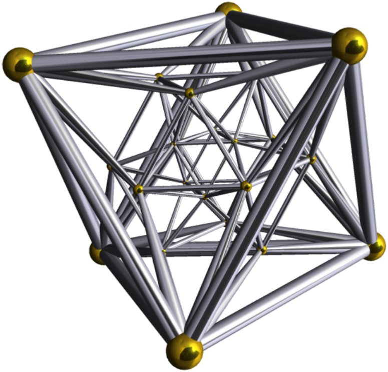


Hexacosichoron



Applause for our heros!

Icositetrachoron



Slides and programs:
http://4d.speicherleck.de/
(sorry for http)

Image sources

Rendered images of four-dimensional bodies created by Robert Webb with his Stella software:
https://en.wikipedia.org/wiki/File:Ortho_solid_011-uniform_polychoron_53p-t0.png
https://en.wikipedia.org/wiki/File:Schlegel_wireframe_5-cell.png
https://en.wikipedia.org/wiki/File:Schlegel_wireframe_8-cell.png
https://en.wikipedia.org/wiki/File:Schlegel_wireframe_16-cell.png
https://en.wikipedia.org/wiki/File:Schlegel_wireframe_24-cell.png
https://en.wikipedia.org/wiki/File:Schlegel_wireframe_120-cell.png
https://en.wikipedia.org/wiki/File:Schlegel_wireframe_600-cell_vertex-centered.png

Image sources

Miscellaneous pictures:
http://4.bp.blogspot.com/_TbkIC-eqFNM/S-dK9dd1cUI/AAAAAAAAFjA/d8qdTHhKy1U/s320/tesseract+unfolded.png
http://gwydir.demon.co.uk/jo/tess/optical6.gif
https://commons.wikimedia.org/wiki/File:Blue_Trefoil_Knot.png
https://en.wikipedia.org/wiki/File:Dodecahedron.svg
https://en.wikipedia.org/wiki/File:Hexahedron.svg
https://en.wikipedia.org/wiki/File:Icosahedron.svg
https://en.wikipedia.org/wiki/File:Octahedron.svg
https://en.wikipedia.org/wiki/File:Tetrahedron.svg
https://mathlesstraveled.files.wordpress.com/2017/01/villarceau-torus-small.jpg
https://upload.wikimedia.org/wikipedia/commons/1/1e/600-cell.gif
https://upload.wikimedia.org/wikipedia/commons/2/24/HC_R1.png
https://upload.wikimedia.org/wikipedia/commons/7/72/Rhombic_dodecahedra_b.png
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http://www.gnuplotting.org/figs/klein_bottle.png
http://www.math.uni-bonn.de/ag/stroppel/Picture_cs2.jpg