– an invitation –

Exploring hypercomputation with the effective topos

Antwerp Logic Adventures
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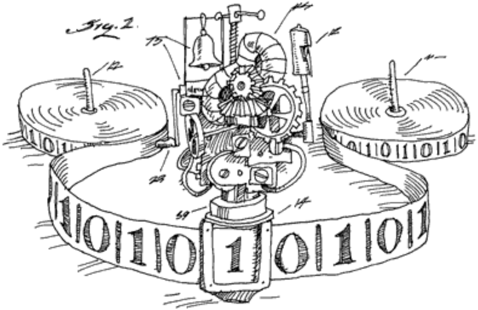
Ingo Blechschmidt
University of Antwerp

Part I

A crash course on ordinal numbers

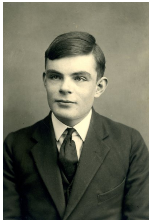
Part II

(Super) Turing machines

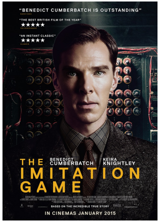


Basics on Turing machines

- Turing machines are idealized computers operating on an **infinite tape** according to a **finite list** of rules.
- The concept is astoundingly robust.
- A subset of $\mathbb{N}$ is **enumerable by a Turing machine** if and only if it is a Σ_1-set.



Alan Turing
(* 1912, † 1954)



worth watching

Alison Bechdel
(* 1960)

Super Turing machines

With super Turing machines, the time axis is more interesting:

- normal: $0,\ 1,\ 2,\ \ldots$
- super: $0,\ 1,\ 2,\ \ldots,\ \omega,\ \omega+1,\ \ldots,\ \omega\cdot 2,\ \omega\cdot 2+1,\ \ldots\ldots\ldots\ldots\ldots\ldots$

On reaching a limit ordinal time step like ω or $\omega \cdot 2, \ldots$

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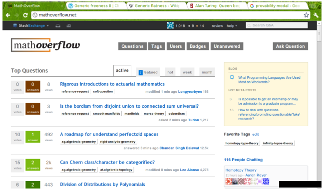
On reaching a limit ordinal time step like ω or $\omega \cdot 2, \ldots$

- the machine is put into a designated state,
- the read/write head is moved to the start of the tape, and
- the tape is set to the "lim sup" of all its previous contents.

Joel David Hamkins



MathOverflow

Andy Lewis

A question for you

What's the behaviour of this super Turing machine?

In the start state and the limit state, check whether the current cell contains a "1".

- If yes, then stop.
- If not, then flash that cell: set it to "1", then reset it to "0". Then unremittingly move the head rightwards.

A question for you

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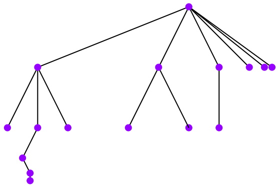
- If yes, then stop.
- If not, then flash that cell: set it to "1", then reset it to "0". Then unremittingly move the head rightwards.

This machine halts after time step ω^2.

Super Turing machines can break out of (some kinds of) infinite loops.

What can super Turing machines do?

- Everything ordinary Turing machines can do.
- Verify number-theoretic statements.
- Decide whether a given ordinary Turing machine halts.
- Simulate super Turing machines.
- Decide Π^1_1- and Σ^1_1-statements:
  - “For every function $\mathbb{N} \to \mathbb{N}$ it holds that …”
  - “There is a function $\mathbb{N} \to \mathbb{N}$ such that …”



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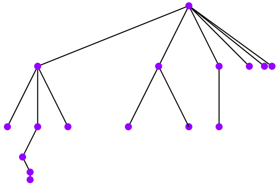
But: Super Turing machines can't compute all functions and can't write every 0/1-sequence to the tape.



Fun facts

- Every super Turing machine either halts or gets caught in an unbreakable infinite loop after **countably many steps**.
- An ordinal number α is **clockable** iff there is a super Turing machine which halts precisely after time step α.
  - Speed-up Lemma: If $\alpha + n$ is clockable, then so is α.
  - Big Gaps Theorem
  - Many Gaps Theorem
  - Gapless Blocks Theorem
- **Lost Melody Theorem:** There are 0/1-sequences which a super Turing machine can recognize, but not write to the tape.

Part III

The effective topos

The effective topos

	statement	in Set	in Eff(TM)	in Eff(STM)
1	Every number is prime or not prime.	✓ (trivially)	✓	✓
2	Beyond every number there is a prime.	✓	✓	✓
3	Every map $\mathbb{N} \to \mathbb{N}$ has a zero or not.	✓ (trivially)	✗	✓
4	Every map $\mathbb{N} \to \mathbb{N}$ is computable.	✗	?	?
5	Every map $\mathbb{R} \to \mathbb{R}$ is continuous.	✗	?	?
6	Markov's principle holds.	✓ (trivially)	?	?
7	Countable choice holds.	✓	?	?
8	There is an injection $\mathbb{R} \hookrightarrow \mathbb{N}$.	✗	?	?

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5 Every map $\mathbb{R} \to \mathbb{R}$ is continuous.	✗	?	?
6 Markov's principle holds.	✓ (trivially)	?	?
7 Countable choice holds.	✓	?	?
8 There is an injection $\mathbb{R} \hookrightarrow \mathbb{N}$.	✗	?	?

"Eff(TM) $\models$ 1" amounts to: There is a Turing machine which determines of any given number whether it is prime or not.

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	statement	in Set	in Eff(TM)	in Eff(STM)
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2	Beyond every number there is a prime.	✓	✓	✓
3	Every map $\mathbb{N} \to \mathbb{N}$ has a zero or not.	✓ (trivially)	✗	✓
4	Every map $\mathbb{N} \to \mathbb{N}$ is computable.	✗	?	?
5	Every map $\mathbb{R} \to \mathbb{R}$ is continuous.	✗	?	?
6	Markov's principle holds.	✓ (trivially)	?	?
7	Countable choice holds.	✓	?	?
8	There is an injection $\mathbb{R} \hookrightarrow \mathbb{N}$.	✗	?	?

"Eff(TM) $\models$ 2" amounts to: There is a Turing machine which, given a number n, computes a prime larger than n.

The effective topos

	statement	in Set	in Eff(TM)	in Eff(STM)
1	Every number is prime or not prime.	✓ (trivially)	✓	✓
2	Beyond every number there is a prime.	✓	✓	✓
3	Every map $\mathbb{N} \to \mathbb{N}$ has a zero or not.	✓ (trivially)	✗	✓
4	Every map $\mathbb{N} \to \mathbb{N}$ is computable.	✗	?	?
5	Every map $\mathbb{R} \to \mathbb{R}$ is continuous.	✗	?	?
6	Markov's principle holds.	✓ (trivially)	?	?
7	Countable choice holds.	✓	?	?
8	There is an injection $\mathbb{R} \hookrightarrow \mathbb{N}$.	✗	?	?

"Eff(TM) $\models$ 3" amounts to: There is a Turing machine which, given a Turing machine computing a map $f : \mathbb{N} \to \mathbb{N}$, determines whether f has a zero or not.

The effective topos

	statement	in Set	in Eff(TM)	in Eff(STM)
1	Every number is prime or not prime.	✓ (trivially)	✓	✓
2	Beyond every number there is a prime.	✓	✓	✓
3	Every map $\mathbb{N} \to \mathbb{N}$ has a zero or not.	✓ (trivially)	✗	✓
4	Every map $\mathbb{N} \to \mathbb{N}$ is computable.	✗	✓ (trivially)	?
5	Every map $\mathbb{R} \to \mathbb{R}$ is continuous.	✗	?	?
6	Markov's principle holds.	✓ (trivially)	?	?
7	Countable choice holds.	✓	?	?
8	There is an injection $\mathbb{R} \hookrightarrow \mathbb{N}$.	✗	?	?

"$\mathrm{Eff}(\mathrm{TM}) \models$ 4" amounts to: There is a Turing machine which, given a Turing machine computing a map $f : \mathbb{N} \to \mathbb{N}$, outputs a Turing machine computing f.

The effective topos

	statement	in Set	in Eff(TM)	in Eff(STM)
1	Every number is prime or not prime.	✓ (trivially)	✓	✓
2	Beyond every number there is a prime.	✓	✓	✓
3	Every map $\mathbb{N} \to \mathbb{N}$ has a zero or not.	✓ (trivially)	✗	✓
4	Every map $\mathbb{N} \to \mathbb{N}$ is computable.	✗	✓ (trivially)	✗
5	Every map $\mathbb{R} \to \mathbb{R}$ is continuous.	✗	?	?
6	Markov's principle holds.	✓ (trivially)	?	?
7	Countable choice holds.	✓	?	?
8	There is an injection $\mathbb{R} \hookrightarrow \mathbb{N}$.	✗	?	?

"$\mathrm{Eff}(\mathrm{STM}) \models$ 4" amounts to: There is a super Turing machine which, given a *super* Turing machine computing a map $f : \mathbb{N} \to \mathbb{N}$, outputs an (*ordinary*) Turing machine computing f.

The effective topos

	statement	in Set	in Eff(TM)	in Eff(STM)
1	Every number is prime or not prime.	✓ (trivially)	✓	✓
2	Beyond every number there is a prime.	✓	✓	✓
3	Every map $\mathbb{N} \to \mathbb{N}$ has a zero or not.	✓ (trivially)	✗	✓
4	Every map $\mathbb{N} \to \mathbb{N}$ is computable.	✗	✓ (trivially)	✗
5	Every map $\mathbb{R} \to \mathbb{R}$ is continuous.	✗	✓ (if MP)	?
6	Markov's principle holds.	✓ (trivially)	?	?
7	Countable choice holds.	✓	?	?
8	There is an injection $\mathbb{R} \hookrightarrow \mathbb{N}$.	✗	?	?

A real number of Eff(TM) is externally represented by a Turing machine M which on input n outputs a rational approximation $M(n)$. These approximations need to be compatible in that $|M(n) - M(m)| \leq 2^{-n} + 2^{-m}$ for all n, m.

Two such machines M and M' represent the same real number iff $|M(n) - M'(m)| \leq 2^{-n} + 2^{-m}$ for all n, m.

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3	Every map $\mathbb{N} \to \mathbb{N}$ has a zero or not.	✓ (trivially)	✗	✓
4	Every map $\mathbb{N} \to \mathbb{N}$ is computable.	✗	✓ (trivially)	✗
5	Every map $\mathbb{R} \to \mathbb{R}$ is continuous.	✗	✓ (if MP)	✗
6	Markov's principle holds.	✓ (trivially)	?	?
7	Countable choice holds.	✓	?	?
8	There is an injection $\mathbb{R} \hookrightarrow \mathbb{N}$.	✗	?	?

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4	Every map $\mathbb{N} \to \mathbb{N}$ is computable.	✗	✓ (trivially)	✗
5	Every map $\mathbb{R} \to \mathbb{R}$ is continuous.	✗	✓ (if MP)	✗
6	Markov's principle holds.	✓ (trivially)	✓ (if MP)	?
7	Countable choice holds.	✓	?	?
8	There is an injection $\mathbb{R} \hookrightarrow \mathbb{N}$.	✗	?	?

Markov's principle states: $\forall f : \mathbb{N} \to \mathbb{N}.\ \neg\neg(\exists n \in \mathbb{N}.\ f(n) = 0) \Rightarrow (\exists n \in \mathbb{N}.\ f(n) = 0)$.

"Eff(TM) $\models$ 6" amounts to: There is a Turing machine which, given a Turing machine computing a map $f : \mathbb{N} \to \mathbb{N}$ and given the promise that it is *not not* the case that f has a zero, determines a zero of f.

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	statement	in Set	in Eff(TM)	in Eff(STM)
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2	Beyond every number there is a prime.	✓	✓	✓
3	Every map $\mathbb{N} \to \mathbb{N}$ has a zero or not.	✓ (trivially)	✗	✓
4	Every map $\mathbb{N} \to \mathbb{N}$ is computable.	✗	✓ (trivially)	✗
5	Every map $\mathbb{R} \to \mathbb{R}$ is continuous.	✗	✓ (if MP)	✗
6	Markov's principle holds.	✓ (trivially)	✓ (if MP)	✓ (if MP)
7	Countable choice holds.	✓	?	?
8	There is an injection $\mathbb{R} \hookrightarrow \mathbb{N}$.	✗	?	?

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"$\mathrm{Eff}(\mathrm{TM}) \models$ 6" amounts to: There is a Turing machine which, given a Turing machine computing a map $f : \mathbb{N} \to \mathbb{N}$ and given the promise that it is *not not* the case that f has a zero, determines a zero of f.

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2	Beyond every number there is a prime.	✓	✓	✓
3	Every map $\mathbb{N} \to \mathbb{N}$ has a zero or not.	✓ (trivially)	✗	✓
4	Every map $\mathbb{N} \to \mathbb{N}$ is computable.	✗	✓ (trivially)	✗
5	Every map $\mathbb{R} \to \mathbb{R}$ is continuous.	✗	✓ (if MP)	✗
6	Markov's principle holds.	✓ (trivially)	✓ (if MP)	✓ (if MP)
7	Countable choice holds.	✓	✓ (always!)	?
8	There is an injection $\mathbb{R} \hookrightarrow \mathbb{N}$.	✗	?	?

Countable choice states: $(\forall x \in \mathbb{N}.\ \exists y \in A.\ \varphi(x, y)) \Rightarrow (\exists f : \mathbb{N} \to A.\ \forall x \in \mathbb{N}.\ \varphi(x, f(x)))$.

"$\mathrm{Eff}(\mathrm{TM}) \models$ 7" amounts to: There is a Turing machine which, given a Turing machine computing for every $x \in \mathbb{N}$ some $y \in A$ together with a witness of $\varphi(x, y)$, outputs a Turing machine computing a suitable choice function $\mathbb{N} \to A$.

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7	Countable choice holds.	✓	✓ (always!)	✓ (always!)
8	There is an injection $\mathbb{R} \hookrightarrow \mathbb{N}$.	✗	✗	✓

Curious size phenomena

$\mathrm{Eff}(\mathrm{STM}) \models$ "There exists an injection $\mathbb{R} \hookrightarrow \mathbb{N}$."

means:

There is a super Turing machine which inputs the source of a super Turing machine A representing a real number and outputs a natural number $n(A)$ such that $n(A) = n(B)$ if and only if A and B represent the same real.

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This statement is witnessed by following super Turing machine:

> Read the source of a super Turing machine A from the tape. Simulate all super Turing machines in a dovetailing fashion. As soon a machine is found which represents the same real as A, output the index of this machine and halt.

Wrapping up

- Effective toposes are a good vehicle for studying the nature of computation.
- Effective toposes build links between constructive mathematics and programming.
- Toposes allow for curious dream axioms.
- Toposes also have a geometric flavour:
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There is more to mathematics than the standard topos.