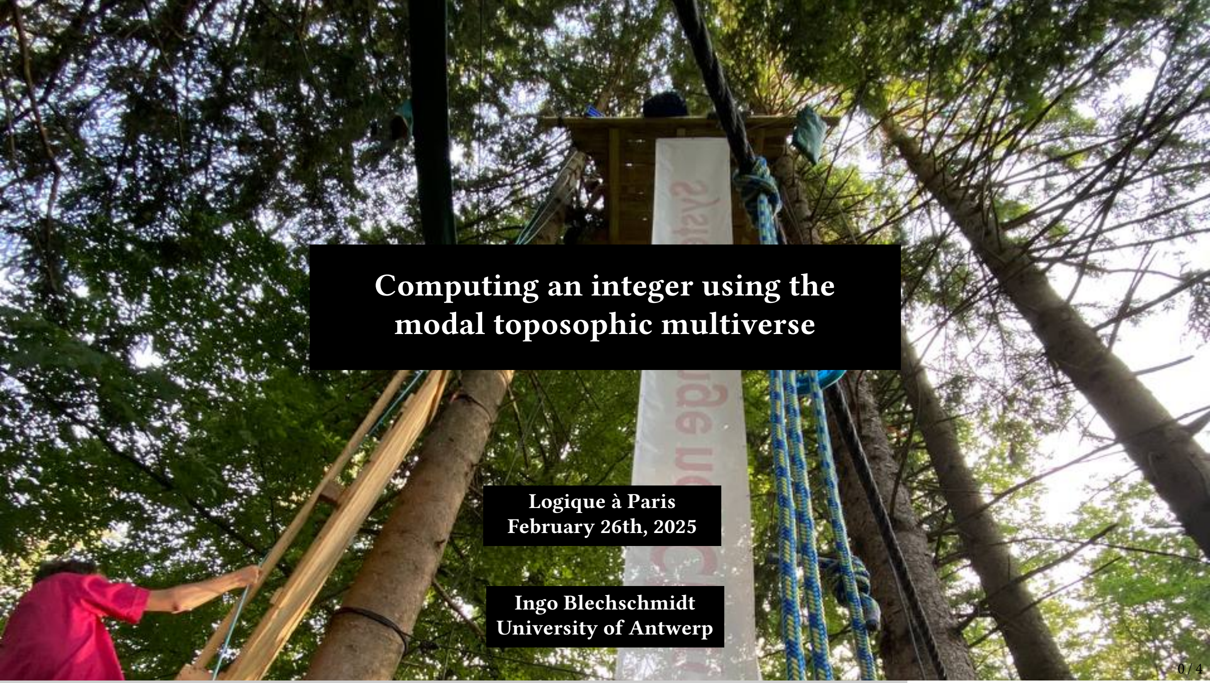

Computing an integer using the modal toposophic multiverse

Logique à Paris
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University of Antwerp

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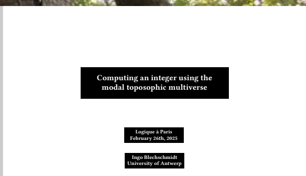
Modal operators for a constructive account of well quasi-orders



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Def. Let $(X, \leq)$ be a quasi-order.

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Well quasi-orders are an important notion in proof theory and termination analysis.

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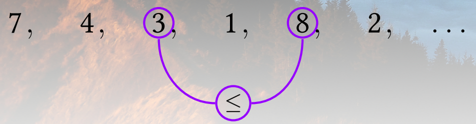
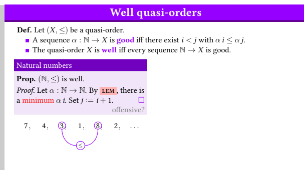
Prop. $(\mathbb{N}, \leq)$ is well.

Proof. Let $\alpha : \mathbb{N} \to \mathbb{N}$. By LEM, there is a minimum $\alpha\, i$. Set $j := i + 1$. □

offensive?





The presented proof rests on the law of excluded middle and hence cannot immediately be interpreted as a program for finding suitable indices $i < j$. However, constructive proofs are also possible (for instance by induction on the value of a given term of the sequence, see Constructive combinatorics of Dickson's Lemma by Iosif Petrakis for several fine quantitative results). And even more: There is a procedure for regarding this proof—and many others in the theory of well quasi-orders—as *blueprints* for more informative constructive proofs. This shall be our motto for today:

Do not take classical proofs literally, instead ask which constructive proofs they are blueprints for.

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Assuming LEM and DC, …

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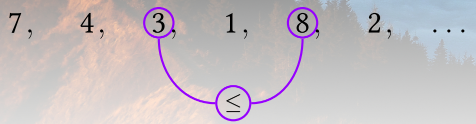

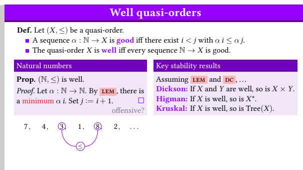

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The displayed stability results, along with several others, provide a flexible toolbox for constructing new well quasi-orders from given ones. However, with the classical formulation of *well*, renamed "well$_\infty$" on the next slide, these results are inherently classical.

In Higman's lemma, the set X^* of finite lists of elements of X is equipped with the following ordering: We have $x_0 \ldots x_{n-1} \leq y_0 \ldots y_{m-1}$ iff there is an increasing injection $f : \{0, \ldots, n-1\} \to \{0, \ldots, m-1\}$ such that $x_i \leq y_{f(i)}$ for all $i < n$.

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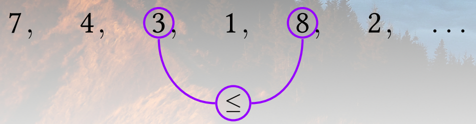

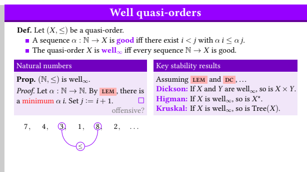

The dependence of the theory on well quasi-orders on classical transfinite methods is already present in one of the first and central observations of this theory:

Lemma. Let X be well$_\infty$. Let $\alpha : \mathbb{N} \to X$. Then there is an infinite increasing subsequence $\alpha\, i_0 \leq \alpha\, i_1 \leq \ldots$.

Proof. Let $K := \{n \in \mathbb{N} \mid \neg \exists m > n.\, \alpha\, n \leq \alpha\, m\}$ be the set of indices of those terms which cannot appear as the first component of a good pair. If K is in bijection with $\mathbb{N}$, there is a subsequence $\alpha\, k_0 \leq \alpha\, k_1 \leq \ldots$ with $k_0, k_1, \ldots \in K$. As X is well$_\infty$, this sequence is good, a contradiction.

Hence K is not in bijection with $\mathbb{N}$. Assuming LEM, it is hence bounded by a number N, and (again with LEM), for every index $a > N$ there is an index $b > a$ such that $\alpha\, a \leq \alpha\, b$. Thus, assuming DC, every number $i_0 > N$ is a suitable starting index for an infinite increasing subsequence. □

The appeal to dependent choice can be removed by always picking the smallest possible next index in $\mathbb{N} \setminus K$, doable by yet another invocation of LEM. But the result remains fundamentally noneffective—in the special case $X = (\{0, 1\}, =)$, the statement of the lemma implies the infinite pigeonhole principle.

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The assertion "Good | [ ]" is pronounced "Good bars the empty list", and is defined as follows: Let B be a predicate on $X^\star$. Then $B \mid \sigma$ is inductively generated by the following two clauses.

1. If $B\sigma$, then $B \mid \sigma$.
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The original notion well$_\infty$:

- ✓ short and simple
- ✓ constructively satisfied for the main examples (but only because of the theory around well$_{\text{ind}}$)
- ✓ concise abstract proofs (albeit employing transfinite methods)
- ✗ main results not constructively attainable
- ✗ philosophically strenuous by the quantification over all sequences
- ✗ not stable under "change of base"—a forcing extension of the universe may well contain more sequences than the base universe
- ✗ negative (universal) condition

The constructive substitute well$_{\text{ind}}$:

- ✓ main results constructive
- ✓ stable under change of base
- ✓ positive (existential) condition
- ✗ proofs intriguing, but also somewhat alien, not just some trivial reshuffling of the classical arguments, classical sequence language cannot be used

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With bar induction, well$_{\text{ind}}$ $\Leftarrow$ well$_\infty$.
Constructively, well$_{\text{ind}}$ $\Rightarrow$ well$_\infty$. Moreover, if X is well$_{\text{ind}}$, then …

- for every *partial* function α, if $\forall n.\, \neg\neg(\alpha\, n\downarrow)$, then $\neg\neg\exists i < j.\, \alpha\, i\downarrow \wedge\, \alpha\, j\downarrow \wedge\, \alpha\, i \leq \alpha\, j$.
- for every *multivalued* function α, $\exists i < j.\, \exists x \in \alpha\, i.\, \exists y \in \alpha\, j.\, x \leq y$.

Constructively, the notion well$_{\text{ind}}$ is much stronger than well$_\infty$, as it ensures goodness (in an appropriate sense) of sequence-like entities which are not actually honest maps $\mathbb{N} \to X$.

For partial maps α, by $\alpha\, n \downarrow$ we mean that α is defined on the input n. If LEM is available, then a partial map such that $\neg\neg(\alpha\, n \downarrow)$ for all $n \in \mathbb{N}$ is already a total map, but without LEM the hypothesis well$_\infty$ does not have anything to say about such a partially-defined sequence.

If DC is available, then every multivalued map contains a singlevalued map, but again without DC the hypothesis well$_\infty$ does not have anything to say about multivalued sequences.

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Central insight: A quasi-order X is well$_{\text{ind}}$ iff $\Box\forall\alpha : \mathbb{N} \to X.\, \exists i < j.\, \alpha\, i \leq \alpha\, j$.

It turns out that these entities are, or give rise to, actual maps $\mathbb{N} \to X$—but in a forcing extension of the universe.

Forcing originated in set theory to construct new models for set theory from given ones, in order to explore the range of set-theoretic possibility. For instance, by forcing we can construct models of zfc validating the continuum hypothesis and also models which falsify it.

We here refer to a simplification of original forcing which is useful in a constructive metatheory. At its core, every forcing extension is just a formula and proof translation of a certain form. For instance, there is a forcing extension validating LEM even if the base universe does not; this forcing extension is not a deep mystery, for a statement holds in that forcing extension iff its double negation translation holds in the base universe and it is well-known that the double negation translation of LEM is an intuitionistic tautology.

Here is a set of slides on constructive forcing, and Section 4 of this joint paper with Peter Schuster contains a written summary of constructive forcing.

The modal multiverse of constructive forcing

Def. A statement φ holds …

- **everywhere** ($\Box\varphi$) iff it holds **in every topos** (over the current base).
- **somewhere** ($\Diamond\varphi$) iff it holds **in some positive topos**.
- **proximally** ($\diamondsuit\!\!\!\cdot\,\varphi$) iff it holds **in some positive overt topos**.

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By *topos*, we mean *Grothendieck topos.* In constructive forcing, a “forcing extension of the base universe” is exactly the same thing as a Grothendieck topos.

A particular member of the rich and varied landscape of toposes is the *trivial topos*, in which every statement whatsoever holds. By restricting to positive toposes, we exclude this special case.

For positive toposes $\mathcal{E}$, a geometric implication holds in $\mathcal{E}$ iff it holds in the base universe. For positive overt toposes $\mathcal{E}$, we even have that a bounded first-order formula holds in $\mathcal{E}$ iff it holds in the base. Hence, for the purpose of verifying a bounded first-order assertion about the base, we can freely pass to a positive overt topos with problem-adapted better higher-order properties (such as that some uncountable set from the base now appears countable, or that an infinite sequence whose existence is predicted by failing dependent choice now actually exists).

Here is a rough early draft of a preprint with more details about the modal multiverse.

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Examples for toposes.

- Set, the category of sets and maps.
- The category of sets and maps which are **defined up to $\neg\neg$**.
- Set$[G]$, the extension obtained by adding a **generic filter** of a **forcing notion** (a quasi-order equipped with a coverage).

The following are not toposes:

- The category of sets and partially defined maps.
- The category of abelian groups.

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Def. A statement φ holds …
everywhere ($\Box\varphi$) iff it holds in every topos (over the current base).
somewhere ($\Diamond\varphi$) iff it holds in some positive topos.
proximally ($\diamonddiamond\varphi$) iff it holds in some positive overt topos.
Def. A (Grothendieck) topos is a category equivalent to the category of sheaves over a small site.
Examples for toposes.
Set, the category of sets and maps.
The category of sets and maps which are defined up to $\neg\neg$.
Set$[G]$, the extension obtained by adding a generic filter of a forcing notion (a quasi-order equipped with a coverage).
The following are not toposes:
The category of sets and partially defined maps.
The category of abelian groups.

The idea to study the modal multiverse of toposes in a principled manner was proposed by Alexander Oldenziel in 2016. *Foreshadowed by:*

1984 André Joyal, Miles Tierney. "An extension of the Galois theory of Grothendieck".

1987 Andreas Blass. "Well-ordering and induction in intuitionistic logic and topoi".

2010s Milly Maietti, Steve Vickers. Ongoing work on arithmetic universes.

2011 Joel David Hamkins. "The set-theoretic multiverse".

2013 Shawn Henry. "Classifying topoi and preservation of higher order logic by geometric morphisms".

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Def. A (Grothendieck) **topos** is a category equivalent to the category of sheaves over a small site.

Multiversal yoga:

1. A quasiorder is well$_{\text{ind}}$ iff *everywhere*, every sequence is good.
2. A ring element is nilpotent iff
   all prime ideals *everywhere* contain it.
3. For every inhabited set X,
   proximally there exists an enumeration $\mathbb{N} \twoheadrightarrow X$.
4. For every ring, *proximally* there exists a maximal ideal.
5. *Somewhere*, the law of excluded middle holds.

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Multiversal yoga:
1 A quasiorder is well$_{\text{ind}}$ iff everywhere, every sequence is good.
2 A ring element is nilpotent iff all prime ideals everywhere contain it.
3 For every inhabited set X, proximally there exists an enumeration $\mathbb{N} \twoheadrightarrow X$.
4 For every ring, proximally there exists a maximal ideal.
5 Somewhere, the law of excluded middle holds.

With the modal language we seek to provide an accessible and modular framework for constructivization results.

For instance, conservativity of classical logic over intuitionistic logic for geometric implications (known under various names such as Barr's theorem, Friedman's trick, escaping the continuation monad, ...) is packaged up by the observation that *somewhere*, the law of excluded middle holds.

Another example: In the community around Krull's lemma, it is well-known that we can constructively infer that a given ring element $x \in A$ is nilpotent from knowing that it is contained in the *generic prime ideal* of A. This entity is not actually an honest prime ideal of the ring A in the base universe, but a certain combinatorial notion (efficiently dealt with using *entailment relations*). Constructive forcing allows us to reify the generic prime ideal as an actual prime ideal in a suitable forcing extension, so in a suitable topos (der little Zariski topos of the ring).

Multiversal constructive combinatorics

Prop. Let X and Y be well$_{\text{ind}}$ quasi-orders. Then $X \times Y$ is well$_{\text{ind}}$.

Multiversal constructive proof. Let $\alpha = (\beta, \gamma) : \mathbb{N} \to X \times Y$ be a sequence in an arbitrary topos. We need to show that α is good, i. e. find indices $n < m$ such that

$$\beta\, n \leq \beta\, m \quad \text{and} \quad \gamma\, n \leq \gamma\, m.$$

It suffices to prove that *somewhere*, α is good, as goodness is a geometric implication (in fact even a geometric formula). Hence without loss of generality, we may suppose LEM.

Thus there is an infinite increasing subsequence

$$\beta\, k_0 \leq \beta\, k_1 \leq \dots .$$

As Y is well$_{\text{ind}}$, the sequence $(\gamma\, k_0, \gamma\, k_1, \dots)$ is good, so there exist $i < j$ with $\gamma\, k_i \leq \gamma\, k_j$. Since we also have $\beta\, k_i \leq \beta\, k_j$, we are done. □

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The displayed multiversal proof closely mimics the classical proof (for well$_\infty$), but is fully constructive (for well$_{\text{ind}}$). It would be possible to streamline this proof and unroll the topos-theoretic machinery, to obtain an explicit algorithm of type

$$\text{Good}\ |_X\ [\,] \times \text{Good}\ |_Y\ [\,] \longrightarrow \text{Good}\ |_{X \times Y}\ [\,].$$

The modal language was recently used to answer a question by Stefano Berardi, Gabriele Buriola and Peter Schuster, see this set of slides.

Multiversal constructive algebra

Thm. Let M be a surjective matrix with more rows than columns over a ring A. Then $1 = 0$ in A.

Classical proof. Assume not. Then there is a maximal ideal $\mathfrak{m}$. The matrix M is surjective over $A/\mathfrak{m}$. Since $A/\mathfrak{m}$ is a field, this is a contradiction to basic linear algebra. □

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The displayed classical proof is quite efficient from the point of view of organizing mathematical knowledge, as it quickly reduces the general situation of dealing with an arbitrary ring to dealing with a field. Alas, read literally, it is hopeless ineffective.

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Unrolled constructive proof (special case). Write $M = \binom{x}{y}$. By surjectivity, have u, v with

$$u \binom{x}{y} = \binom{1}{0} \quad \text{and} \quad v \binom{x}{y} = \binom{0}{1}.$$

Hence $1 = (vy)(ux) = (uy)(vx) = 0$. □

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Backup slides.

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Ingredients for forcing

To construct a forcing extension, we require:

1. a base universe V
2. a preorder L of **forcing conditions** in V, pictured as **finite approximations** (*convention:* $\tau \preccurlyeq \sigma$ means that τ is a better finite approximation than σ)
3. a **covering system** governing how finite approximations evolve to better ones (for each $\sigma \in L$, a set $\mathrm{Cov}(\sigma) \subseteq P(\downarrow\sigma)$, with a simulation condition)

In the forcing extension V^{∇}, there will then be a **generic filter** (ideal object).

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For the generic surjection $\mathbb{N} \twoheadrightarrow X$

Use **finite lists** $\sigma \in X^*$ as forcing conditions, where $\tau \preccurlyeq \sigma$ iff σ is an initial segment of τ, and be prepared to grow σ to …

(a) one of $\{\sigma x \mid x \in X\}$, to make σ more defined

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For the generic prime ideal of a ring A

Use **f.g. ideals** as forcing conditions, where $\mathfrak{b} \preccurlyeq \mathfrak{a}$ iff $\mathfrak{b} \supseteq \mathfrak{a}$, and be prepared to grow $\mathfrak{a}$ to ...

(a) one of $\emptyset$, if $1 \in \mathfrak{a}$, to make $\mathfrak{a}$ more proper

(b) one of $\{\mathfrak{a} + (x), \mathfrak{a} + (y)\}$, if $xy \in \mathfrak{a}$, to make $\mathfrak{a}$ more prime

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The eventually monad

Let L be a forcing notion.

Let P be a monotone predicate on L (if $\tau \preccurlyeq \sigma$, then $P\sigma \Rightarrow P\tau$).
For instance, in the case $L = X^*$:

- Repeats $x_0 \dots x_{n-1} :\equiv \exists i.\, \exists j.\, i < j \wedge x_i = x_j$
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We then define "$P \mid \sigma$" ("P bars σ") inductively by the following clauses:

1. If $P\sigma$, then $P \mid \sigma$.
2. If $P \mid \tau$ for all $\tau \in R$, where R is some covering of σ, then $P \mid \sigma$.

So $P \mid \sigma$ expresses in a **direct inductive fashion**:

> "No matter how σ evolves to a better approximation τ, eventually $P\tau$ will hold."

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Proof translations

Thm. Every IQC-proof remains correct, with at most a polynomial increase in length, if throughout we replace

$$\begin{array}{ccccrcl} \exists & \rightsquigarrow & \exists^{\mathrm{cl}}, & \text{where} & \exists^{\mathrm{cl}} & :\equiv & \neg\neg\exists, \\ \vee & \rightsquigarrow & \vee^{\mathrm{cl}}, & \text{where} & \alpha \vee^{\mathrm{cl}} \beta & :\equiv & \neg\neg(\alpha \vee \beta), \\ = & \rightsquigarrow & =^{\mathrm{cl}}, & \text{where} & s =^{\mathrm{cl}} t & :\equiv & \neg\neg(s = t). \end{array}$$

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Similarly for arbitrary forcing extensions V^{∇}, "just with ∇ instead of $\neg\neg$".

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∃ ⇝ ∃cl, where ∃cl :≡ ¬¬∃,
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Ex. As $\neg\neg(\varphi \vee \neg\varphi)$ is a theorem of IQC, the law of excluded middle holds in $V^{\neg\neg}$.

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The ∇-translation

For bounded first-order formulas over the (large) first-order signature which has

1. one sort $\underline{X}$ for each set X in the base universe,
2. one n-ary function symbol $\underline{f} : \underline{X_1} \times \cdots \times \underline{X_n} \to \underline{Y}$ for each map $f : X_1 \times \cdots \times X_n \to Y$,
3. one n-ary relation symbol $\underline{R} \hookrightarrow \underline{X_1} \times \cdots \times \underline{X_n}$ for each relation $R \subseteq X_1 \times \cdots \times X_n$, and
4. an additional unary relation symbol $G \hookrightarrow \underline{L}$ (for the *generic filter* of L),

we recursively define:

$\sigma \models s = t$	iff	$\nabla\sigma.\, [\![s]\!] = [\![t]\!]$.	$\sigma \models \underline{R}(s_1, \ldots, s_n)$	iff	$\nabla\sigma.\, R([\![s_1]\!], \ldots, [\![s_n]\!])$.
$\sigma \models \varphi \Rightarrow \psi$	iff	$\forall(\tau \preccurlyeq \sigma).\, (\tau \models \varphi) \Rightarrow (\tau \models \psi)$.	$\sigma \models G\tau$	iff	$\nabla\sigma.\, \sigma \preccurlyeq [\![\tau]\!]$.
$\sigma \models \top$	iff	$\top$.	$\sigma \models \bot$	iff	$\nabla\sigma.\, \bot$
$\sigma \models \varphi \wedge \psi$	iff	$(\sigma \models \varphi) \wedge (\sigma \models \psi)$.	$\sigma \models \varphi \vee \psi$	iff	$\nabla\sigma.\, (\sigma \models \varphi) \vee (\sigma \models \psi)$.
$\sigma \models \forall(x : \underline{X}).\, \varphi$	iff	$\forall(\tau \preccurlyeq \sigma).\, \forall(x_0 \in X).\, \tau \models \varphi[\underline{x_0}/x]$.	$\sigma \models \exists(x : \underline{X}).\, \varphi$	iff	$\nabla\sigma.\, \exists(x_0 \in X).\, \sigma \models \varphi[\underline{x_0}/x]$.

Finally, we say that φ "holds in V^∇" iff for all $\sigma \in L$, $\sigma \models \varphi$.

forcing notion	statement about V^∇	external meaning
surjection $\mathbb{N} \twoheadrightarrow X$	"the gen. surj. is surjective"	$\forall(\sigma \in X^*).\, \forall(a \in X).\, \nabla(\tau \preccurlyeq \sigma).\, \exists(n \in \mathbb{N}).\, \tau[n] = a$.

The ∇-translation

$\sigma \models s = t$	iff	$\nabla\sigma.\ [\![s]\!] = [\![t]\!]$.	$\sigma \models \underline{R}(s_1, \ldots, s_n)$	iff	$\nabla\sigma.\ R([\![s_1]\!], \ldots, [\![s_n]\!])$.
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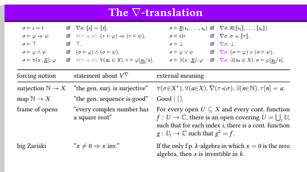
map $\mathbb{N} \to X$	"the gen. sequence is good"	Good $\mid$ [ ].
frame of opens	"every complex number has a square root"	For every open $U \subseteq X$ and every cont. function $f : U \to \mathbb{C}$, there is an open covering $U = \bigcup_i U_i$ such that for each index i, there is a cont. function $g : U_i \to \mathbb{C}$ such that $g^2 = f$.
big Zariski	"$x \neq 0 \Rightarrow x$ inv."	If the only f.p. k-algebra in which $x = 0$ is the zero algebra, then x is invertible in k.



The ∇-translation

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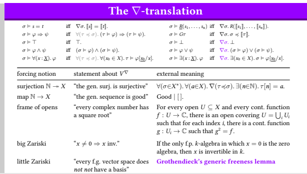
big Zariski	"$x \neq 0 \Rightarrow x$ inv."	If the only f.p. k-algebra in which $x = 0$ is the zero algebra, then x is invertible in k.
little Zariski	"every f.g. vector space does *not not* have a basis"	**Grothendieck's generic freeness lemma**



Outlook

Passing to and from extensions

Thm. Let φ be a **bounded first-order formula** not mentioning G. In each of the following situations, we have that φ holds in V^∇ iff φ holds in V:

1. L and all coverings are inhabited (proximality).
2. L contains a top element, every covering of the top element is inhabited, and φ is a coherent implication (positivity).

The mystery of nongeometric sequents

The **generic ideal** of a ring is maximal:

$$(x \in \mathfrak{a} \Rightarrow 1 \in \mathfrak{a}) \Longrightarrow 1 \in \mathfrak{a} + (x).$$

The **generic ring** is a field:

$$(x = 0 \Rightarrow 1 = 0) \Longrightarrow (\exists y.\, xy = 1).$$

Traveling the multiverse …

LEM is a **switch** and **holds positively**; being countable is a **button**.

Every instance of DC **holds proximally**.

A geometric implication is provable iff it holds **everywhere**.

… upwards, but always keeping ties to the base.

2025-02-27

Modal operators for a constructive account of well quasi-orders
└─Basics of forcing
└─Outlook

More on forcing notions

Def. A **forcing notion** consists of a preorder L of **forcing conditions**, and for every $\sigma \in L$, a set $\mathrm{Cov}(\sigma) \subseteq P(\downarrow\sigma)$ of **coverings** of σ such that: If $\tau \preccurlyeq \sigma$ and $R \in \mathrm{Cov}(\sigma)$, there should be a covering $S \in \mathrm{Cov}(\tau)$ such that $S \subseteq \downarrow R$.

	preorder L	coverings of an element $\sigma \in L$	filters of L
1	X^*	$\{\sigma x \mid x \in X\}$	maps $\mathbb{N} \to X$
2	X^*	$\{\sigma x \mid x \in X\}$, $\{\sigma\tau \mid \tau \in X^*, a \in \sigma\tau\}$ for each $a \in X$	surjections $\mathbb{N} \twoheadrightarrow X$
3	f.g. ideals	—	ideals
4	f.g. ideals	$\{\sigma + (a), \sigma + (b)\}$ for each $ab \in \sigma$, $\{\}$ if $1 \in \sigma$	prime ideals
5	opens	$\mathcal{U}$ such that $\sigma = \bigcup \mathcal{U}$	points
6	$\{\star\}$	$\{\star \mid \varphi\} \cup \{\star \mid \neg\varphi\}$	witnesses of LEM

Def. A *filter* of a forcing notion (L, Cov) is a subset $F \subseteq L$ such that

1. F is upward-closed: if $\tau \preccurlyeq \sigma$ and if $\tau \in F$, then $\sigma \in F$;
2. F is downward-directed: F is inhabited, and if $\alpha, \beta \in F$, then there is a common refinement $\sigma \preccurlyeq \alpha, \beta$ such that $\sigma \in F$; and
3. F splits the covering system: if $\sigma \in F$ and $R \in \mathrm{Cov}(\sigma)$, then $\tau \in F$ for some $\tau \in R$.

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