A primer to the ♥ set-theoretic multiverse philosophy ♥

– an invitation –

36th Chaos Communication Congress

Questions are very much welcome! Please interrupt me mid-sentence.

Ingo Blechschmidt

Prehistory

☹

Inconsistency

at the heart of mathematics ~ 1900

“Let R be the set of all those sets which don’t contain themselves.”

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Formal proofs as the gold standard
to judge correctness ~ 1920

Zermelo–Fraenkel set theory with the axiom of choice, ZFC

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Zermelo–Fraenkel set theory with the axiom of choice, ZFC

☹

Incompleteness ∼ 1931

An outline of ZFC

Axioms of ZFC include:

1. $\exists x.\ \forall y.\ \neg(y \in x)$
   "There is a set x such that for every set y, it is not the case that y is an element of x."
2. $\forall x.\ \forall y.\ \exists w.\ (x \in w \wedge y \in w)$
   "Given any sets x and y, there is a set which contains x and y."
3. $\forall x.\ \exists z.\ \forall w.\ ((\forall y.\ (y \in w \Rightarrow y \in x)) \Rightarrow w \in z)$.
   "Given any set x, there is a set which contains all subsets of x."

Inference rules of ZFC include:

1. From $\varphi \wedge \psi$ deduce φ.
2. Modus ponens: From φ and $\varphi \Rightarrow \psi$ deduce ψ.
3. Law of excluded middle: Deduce $\varphi \vee \neg\varphi$.

Def. "ZFC $\vdash \varphi$" means: There is a ZFC-proof of φ.

Ex. ZFC $\vdash 2 + 2 = 4$, ZFC $\vdash 2 + 2 \neq 5$, ZFC $\nvdash 2 + 2 = 5$.

Foundational incompleteness

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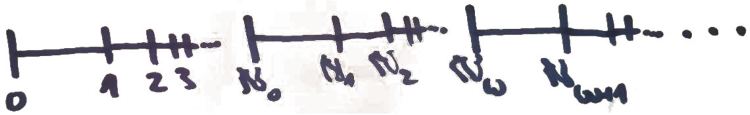
Examples for statements which are **independent from ZFC**:

1. "The system ZFC is consistent."
2. "The value of BB(1919) is $1 + \cdots + 1$.", where the sum comprises BB(1919) summands.
3. "A certain specific equation does not have a solution."
4. "The continuum hypothesis holds."

The continuum hypothesis and its traditional dream solution



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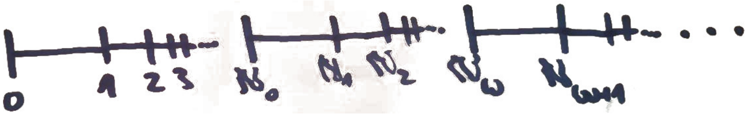

all natural numbers		all integers
0	$\longleftrightarrow$	0
1	$\longleftrightarrow$	1
2	$\longleftrightarrow$	-1
3	$\longleftrightarrow$	2
4	$\longleftrightarrow$	-2
5	$\longleftrightarrow$	3
6	$\longleftrightarrow$	-3
$\vdots$		$\vdots$

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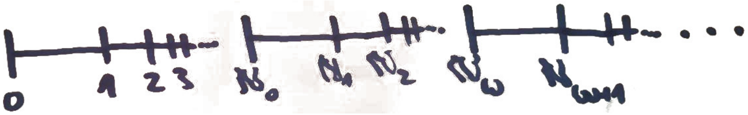

Def. The size of the set $\mathbb{R}$ of real numbers is $\mathfrak{c}$.

Thm. (Cantor's diagonal argument). $\mathfrak{c} > \aleph_0$ (in fact, $\mathfrak{c} = 2^{\aleph_0}$).

Where is $\mathfrak{c}$ located on the cardinal number line?

The **continuum hypothesis** CH states: "$\mathfrak{c} = \aleph_1$." Alas:

1. If there is a ZFC-proof of ¬CH, then ZFC is inconsistent. [Gödel 1938]
2. If there is a ZFC-proof of CH, then ZFC is inconsistent. [Cohen 1963]

Hence the traditional dream solution:
Devise additional axioms to settle CH.

Models of ZFC

Def. A **model** of ZFC is a collection M together with a binary relation on M for which the axioms of ZFC are satisfied.

- Elements of M are called "sets of M", "what M believes to be sets" or "M-sets". The relation is written "$\in$".
- "$M \models \varphi$" means: The statement φ holds for M-sets.

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Non-example 1. Let $M = \{\heartsuit\}$ and declare $\heartsuit \in \heartsuit$.
Non-example 2. Let $M = \{\heartsuit\}$ and declare $\heartsuit \notin \heartsuit$.
Example. Let M be the collection of all sets in the platonic heaven. Declare $x \in y$ if and only if x is actually an element of y.

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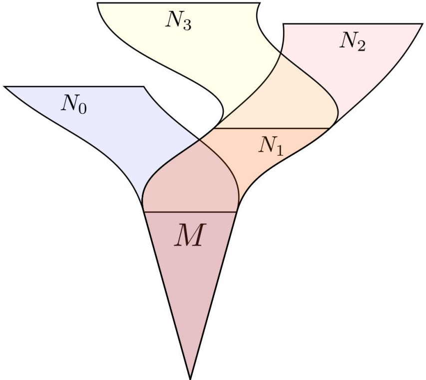
Embrace the **multiverse**,
the entirety of all models of ZFC.

Traveling the multiverse

Def. (modal operators)

1. "$\Diamond\varphi$" means that φ holds in **some** extension of the universe.
2. "$\Box\varphi$" means that φ holds in **every** extension of the universe.



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Def.

1. A **switch** is a statement φ such that $\Box((\Diamond\varphi) \wedge (\Diamond(\neg\varphi)))$:
   No matter where we travel to from the current universe, there will always be a road to a universe in which φ holds and there will always be a road to a universe in which φ does not hold.
2. A **button** is a statement φ such that $\Box(\Diamond(\Box\varphi))$:
   No matter where we travel to from the current universe, there will always be a road to a universe such that φ holds there and in any further universe reachable from there.

Example. CH is a switch.

Notable features of the multiverse

1. The **mirage of uncountability**: Any model is **merely countable** from the point of view of a sufficiently larger model.
2. The **mirage of well-foundedness**: Any model is **ill-founded** from the perspective of an appropriate other model.
3. Some models are maximally rich in that they validate $(\Diamond(\Box\varphi)) \Rightarrow (\Box\varphi)$.
4. There is a certain Turing machine P such that for any function $f : \mathbb{N} \to \mathbb{N}$, there is some model in which P computes f.
5. Set-theoretic geology: A **ground** of a model M is a model M' such that M is an extension of M'. The **mantle** of M is the intersection of all its grounds. … an ancient paradise?